MATH2010B Advanced Calculus I, 2014-15

Midterm Test Solutions

Q1. (a) *(6 points)* Find an equation of the plane passing through the point $(1, 0, 4)$ and perpendicular to the line $L = \{(2, 5, 8) + t(-1, 19, 1) : t \in \mathbb{R}\}$.

Solution: A normal to the plane is $\mathbf{n} = (-1, 19, 1)$ and a point on the plane is $\mathbf{p} = (1, 0, 4)$. The equation for the plane passing through $\mathbf{p}$ and normal to $\mathbf{n}$ is given by

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}.$$

Since $\mathbf{p} \cdot \mathbf{n} = (1, 0, 4) \cdot (-1, 19, 1) = -1 + 0 + 4 = 3$, we get

$$-x + 19y + z = 3.$$

(b) *(6 points)* Let $E = \{(x, y) : x^2 + 4y^2 = 4\}$ be an ellipse. Write down a parametrization $\gamma(t) : [a, b] \to \mathbb{R}^2$ of the ellipse and the definite integral that computes the length of the ellipse E (you DO NOT have to evaluate the integral).

Solution: A parametrization is given by

$$\gamma(t) = (2\cos t, \sin t), \qquad t \in [0, 2\pi].$$

Hence we can calculat

$$\gamma'(t) = (-2\sin t, \cos t),$$

$$\|\gamma'(t) = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{1 + 3\sin^2 t}.$$

Therefore, the length of the ellipse is

$$L = \int_0^{2\pi} \sqrt{1 + 3\sin^2 t}\, dt.$$

Q2. *(8 points)* Define the function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{1}{x}\sin xy & \text{if } x \neq 0, \\ y & \text{if } x = 0. \end{cases}$$

Evaluate the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ or explain why the limit does not exist.

Solution: Note that along different straight lines approaching $(0, 0)$,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x, y) = \lim_{y\to 0} y = 0,$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x, y) = \lim_{x\to 0} \frac{1}{x}\sin(x \cdot 0) = 0,$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} f(x, y) = \lim_{x\to 0} \frac{1}{x}\sin kx^2 = \lim_{x\to 0} kx \cdot \lim_{x\to 0} \frac{\sin kx^2}{kx^2} = 0 \cdot 1 = 0.$$

In fact, we have $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. To see this, recall that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. By definition, there exists $\delta_1 > 0$ small enough such that

$$\left|\frac{\sin\theta}{\theta} - 1\right| < 1 \qquad \text{for any } 0 < |\theta| < \delta_1.$$

This implies that

$$\left|\frac{\sin\theta}{\theta}\right| \le 2 \qquad \text{for any } 0 < |\theta| < \delta_1.$$

To show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, we use the $\epsilon-\delta$-definition of limit. Let $\epsilon > 0$ be a small number, say $\epsilon < 1$, we take $0 < \delta < \min(\sqrt{\delta_1}, \epsilon/2)$, then we want to show that for any (x, y) such that $\sqrt{x^2+y^2} < \delta$, we have

$$|f(x,y)| < \epsilon.$$

Since $f(x,y)$ is defined differently at different points, we have to consider 3 cases:

Case 1: $x \neq 0$ and $y \neq 0$. Then, since $|xy| \le \frac{x^2+y^2}{2} < \frac{\delta_1}{2}$, we have

$$|f(x,y)| = \left|\frac{1}{x}\sin xy\right| = |y|\left|\frac{\sin xy}{xy}\right| \le 2|y| < \epsilon.$$

Case 2: $x = 0$ and $y \neq 0$. Then clearly $|f(x,y)| = |y| < \epsilon$.

Case 3: $x \neq 0$ and $y = 0$. Then $|f(x,y)| = 0 < \epsilon$.

Combining all these 3 cases, we have proved our assertion.

Q3. *(8 points)* Find an equation for the tangent plane of the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = e^y \sin x\}$$

at the point $(\pi, 0, 0)$.

Solution: Let $f(x,y) = e^y \sin x$. Then taking partial derivatives, we get

$$\begin{cases} f_x = e^y \cos x, \\ f_y = e^y \sin x, \end{cases}$$

which implies $f_x(\pi, 0) = -1$ and $f_y(\pi, 0) = 0$. The equation for the tangent plane is given by the formula

$$z = f(\pi,0) + f_x(\pi,0)(x-\pi) + f_y(\pi,0)(y-0).$$

Since $f(\pi, 0) = 0$, the equation is just

$$x + z = \pi.$$

Q4. *(8 points)* Show that the function

$$u(t,x) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

satisfies the partial differential equation $u_t = u_{xx}$ for any $t > 0$ and $x \in \mathbb{R}$.

Solution: Taking partial derivatives directly.

$$\begin{aligned} u_t &= -\frac{1}{2}\frac{1}{t^{3/2}} e^{-\frac{x^2}{4t}} + \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(\frac{x^2}{4t^2}\right) \\ &= \frac{1}{t^{3/2}} e^{-\frac{x^2}{4t}} \left(-\frac{1}{2} + \frac{x^2}{4t}\right). \end{aligned}$$

$$u_x = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(-\frac{x}{2t}\right).$$

$$\begin{aligned} u_{xx} &= \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(-\frac{1}{2t} + \frac{x^2}{4t^2} \right) \\ &= \frac{1}{t^{3/2}} e^{-\frac{x^2}{4t}} \left(-\frac{1}{2} + \frac{x^2}{4t} \right). \end{aligned}$$

Hence, we have shown that $u_t = u_x x$.

Q5. *(12 points)* Find the maximum and minimum of the function $f(x, y) = xy$ on the region

$$R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 2\}.$$

Locate the points where the minimum and maximum are achieved.

Solution: For the interior critical points, we solve

$$\begin{cases} f_x = y = 0 \\ f_y = x = 0 \end{cases}$$

to get only one critical point $(0, 0)$ with $f(0, 0) = 0$.

For the boundary points, we use polar coordinates (r, θ), hence ∂R is simply $r = \sqrt{2}$. In polar coordinates,

$$f(\sqrt{2}, \theta) = 2 \sin\theta \cos\theta = \sin 2\theta,$$

which clearly has its maximum$= 1$ when $\theta = \pi/4$ and $5\pi/4$, and has its minimum$= -1$ when $\theta = 3\pi/4$ and $7\pi/4$.

Combining all these, the maximum of f is 1 located at $(1, 1)$ and $(-1, -1)$ and the minimum of f is -1 located at $(1, -1)$ and $(-1, 1)$.

Q6. *(12 points)* Consider the function

$$f(x, y) = \begin{cases} x - 2y \tan^{-1} \frac{x}{y} & \text{when } y \ne 0 \\ x & \text{when } y = 0 \end{cases}$$

compute the partial derivative f_x and determine if f_x is continuous at $(0, 0)$.

Solution: Differentiating directly, we get

$$f_x(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{when } y \ne 0, \\ 1 & \text{when } y = 0. \end{cases}$$

Note that

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} f_x(x, y) = \lim_{\substack{(x,y)\to(0,0) \\ y=0}} 1 = 1,$$

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} f_x(x, y) = \lim_{y \to 0} \frac{-y^2}{y^2} = -1,$$

which are not equal. Therefore, f_x is NOT continuous at $(0, 0)$.